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ROBUSTNESS OF STABILITY CONDITIONS FOR LINEAR TIME-INVARIANT FEEDBACK SYSTEMS

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Abstract

The robustness of stability conditions for linear time-invariant feedback systems is examined assuming three different types of representations: state-space representation, coprime matrix fraction representation, and transfer function representation. ~~We stress~~ the importance is stressed of certain details of the representation used and, even more, the importance of making sure that the allowed perturbations be relevant to the physical situation under study.

I. Introduction

Engineers design for production: therefore it is required that their nominal design as well as a very high portion of the systems produced — which suffer from element deviations and manufacturing tolerances — meet the specifications. Furthermore, the systems produced must meet the specifications not only as they leave the production line but also in the field — where they suffer from temperature effects, aging, weathering etc.— . Hence the interest in sensitivity and robustness. It is for these reasons that this subject has an extensive literature [e.g. 1,2].

In this paper, we consider the robustness of the stability conditions for a continuous-time, linear, time-invariant, lumped, multi-input multi-output feedback system. (See Fig. 1) If the feedback system is made of an interconnection of stable subsystems, it seems intuitively clear that under some reasonable conditions and under reasonable allowed perturbations the stability conditions are robust. But what if the subsystems are unstable? Might it not happen that due to perturbation some kind of pole-zero cancellation is destroyed? The purpose of this paper is to examine the conditions which, under several representations, guarantee robustness of the stability conditions. We will find that the nature of the representation and the nature of *the allowed perturbations play a crucial role.* This will also lead us to make some remarks on the relevance of some parameter perturbations.

In order to avoid repetitions, we start by defining some terms and some notations.

II. Preliminary Definitions

$\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}(s)$ and $\mathbb{R}[s]$ denote, respectively, the fields of real numbers, of complex numbers, of rational functions with real coefficients and the commutative ring of polynomials with real coefficients. The superscripts "n" and "nxm" (as in $\mathbb{R}^n$, $\mathbb{R}(s)^{nxm}$) denote the corresponding ordered n-tuples and nxm arrays. $\mathbb{C}_+ := \{s \in \mathbb{C} \mid \text{Re } s \geq 0\}$ denotes the closed right-half-plane. $\mathring{\mathbb{C}}_- := \{s \in \mathbb{C} \mid \text{Re } s < 0\}$ denotes the open left-half-plane. Given any scalar rational function, we assume once and for all that it is written as $n(s)/d(s)$ where the polynomials n and d are coprime and d is monic.

A continuous-time, linear, time-invariant, lumped, multi-input multi-output system is said to be exponentially stable (abbr. exp. stable) iff its transfer function $G(s) \in \mathbb{R}(s)^{nxm}$ is proper (i.e. bounded at infinity) and G(s) has no $\mathbb{C}_+$-poles. For example, for the system shown on Fig. 1, this means that $G: (u_1,u_2) \longmapsto (e_1,e_2)$ has these properties.

III. System Description

We consider the input-output stability problem of the continuous-time, linear, time-invariant, lumped, multi-input multi-output feedback system $S: (u_1,u_2) \longmapsto (e_1,e_2)$ described in frequency domain by (see Fig. 1)

$$u_1 = e_1 + G_2 e_2 \qquad u_2 = e_2 - G_1 e_1 \tag{1}$$

where $G_1, G_2 \in \mathbb{R}(s)^{nxn}$, the u_i's are the inputs and the e_i's are the "errors". The transfer function of S is $G: (u_1,u_2) \longmapsto (e_1,e_2)$.† Throughout this this paper, we make the following assumption:

Assumption: The transfer functions G_1, G_2 are proper (i.e. bounded at infinity) and

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†We only need to consider $(u_1,u_2) \longmapsto (e_1,e_2)$ because the map $(u_1,u_2) \longmapsto (y_1,y_2)$ is exp. stable if and only if the map $(u_1,u_2) \longmapsto (e_1,e_2)$ is exp. stable [10].

$$\det[I+G_2(\infty)\, G_1(\infty)] \neq 0. \qquad (2)$$

This assumption implies that the transfer function G of S exists and is proper. Note that a transfer function must be proper in order to have a state-space representation.

We investigate the following question: given that G_1, G_2 are described in a specified way and given some set of allowed perturbations, are the input-output stability conditions for the system S robust under the allowed perturbations in G_1, G_2? By this we mean: if the feedback system S is exp. stable for the nominal values of G_1, G_2, does it imply that it will remain exp. stable for all sufficiently small perturbations in G_1, G_2, selected from the allowed set? It will turn out that the answer depends very much on the representation of G_1, G_2 and the set of allowed perturbations.

IV. A Preliminary Lemma

All the argumentation below hinges on a lemma that essentially says "small perturbations in the coefficients of an algebraic equation cause small perturbations in its zeros." More precisely, we state a well-known lemma.

Lemma [3, Thm 9.17.4]. Let $D(z_i;\varepsilon)$ denotes the open disc in $\mathbb{C}$ centered on z_i and with radius ε. Consider the polynomial p defined by

$$p(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n \qquad (3)$$

where $a_i \in \mathbb{R}$, $\forall i$, and without loss of generality, $a_0 > 0$. Let the polynomial p have q pairwise *distinct zeros:* $z_1, z_2, \ldots, z_q$ with respective multiplicities m_i, (hence $\Sigma m_i = n$). Then for all $\varepsilon > 0$, there is an $\eta(\varepsilon) > 0$ such that for all δa_i satisfying

$$|\delta a_i| < \eta \qquad i = 0,1,\ldots,n \qquad (4)$$

the perturbed polynomial

$$(a_0+\delta a_0)s^n + \ldots + (a_{n-1}+\delta a_{n-1})s + (a_n+\delta a_n) \qquad (5)$$

still has m_i zeros in $D(z_i;\varepsilon)$ for $i = 1,2,\ldots,q$.

Comments: (i) If all zeros of p were simple, this lemma would be a direct consequence of the implicit function theorem; the point is that the continuous dependence of the zeros is still valid in the case of multiple zeros. (ii) It is crucial to observe that the degree of p was not affected by the perturbations: indeed suppose that instead of (5) we had

$$\tilde{p}(s) = \delta a_{-1}s^{n+1} + (a_0+\delta a_0)s^n + \ldots + (a_n+\delta a_n) \qquad (6)$$

as the perturbed polynomial; then, for $\eta > 0$ sufficiently small, if $|\delta a_i| < \eta$ for $i = -1,0,1,\ldots,n$, $\tilde{p}$ would have n+1 zeros, n of them in the discs $D(z_i;\varepsilon)$ and one approximately equal to $-(a_0+\delta a_0)/\delta a_{-1}$. (This approximate zero is the leading term of a sequence of successive approximation which converges for η small).

This additional zero is very large: its sign is positive or negative according to $\delta a_{-1} < 0$ or $\delta a_{-1} > 0$, respectively. Similarly, if there had been several additional terms of degrees larger than n, there would have been several such zeros with very large magnitude and whose location in the s-plane depends on the magnitudes and signs of the δa_{-i}'s. Note that when the perturbed problem is of the form (6), we are essentially dealing with a singular perturbation problem; see e.g. [4,5].

V. Robustness Results

Case I: The transfer functions G_1 and G_2 are specified by minimal state-space representations

It is well known that if [A,B,C,D] is a minimal state-space representation of the proper transfer function G, then G is exp. stable if and only if det(sI-A) has all its zeros in $\mathring{\mathbb{C}}_-$. For $i = 1,2$, let $[A_i,B_i,C_i,D_i]$ be a minimal state-space representation of the proper transfer function G_i with the state $x_i \in \mathbb{R}^{n_i}$. Let [A,B,C,D] be the state-space representation of the feedback system S: $(u_1,u_2) \longmapsto (e_1,e_2)$ with the state (x_1,x_2). We know that [6] (i) [A,B,C,D] is a minimal state-space representation of S, (ii)

$$A = \begin{bmatrix} A_1-B_1(I+D_2D_1)^{-1}D_2C_1 & -B_1(I+D_2D_1)^{-1}C_2 \\ B_2(I+D_1D_2)^{-1}C_1 & A_2-B_2(I+D_1D_2)^{-1}D_1C_2 \end{bmatrix} \qquad (7)$$

and *(iii)*

$$\det(sI-A) = \det(sI-A_1)\cdot\det(sI-A_2)\cdot\det(I+G_2G_1)(s) \qquad (8)$$

Now the feedback system S is exp. stable if and only if det(sI-A) has all its zeros in $\mathring{\mathbb{C}}_-$; furthermore for any perturbation $[\delta A_i,\delta B_i,\delta C_i,\delta D_i]$ in the constant matrices $[A_i,B_i,C_i,D_i]$, $i = 1,2$, the degree of det(sI-A) remains equal to n_1+n_2. In view of (7), we obtain from the lemma above a well-known result:

Robustness Result I.1.

If the given feedback system S is exp. stable at the nominal data point, then for any sufficiently small perturbation $[\delta A_i,\delta B_i,\delta C_i,\delta D_i]$, $i = 1,2$, the resulting perturbed system is still exp. stable. □

Remark I.1: In many applications, neither measurements nor system-component element values directly specify the $[A_i,B_i,C_i,D_i]$; therefore, engineers should ask whether the perturbations $[\delta A_i,\delta B_i,\delta C_i,\delta D_i]$ of this analysis cover all the possible perturbations expected in the contemplated physical environment.

Remark I.2: Note that for sufficiently small allowed perturbations, $[A_i+\delta A_i, B_i+\delta B_i, C_i+\delta C_i, D_i+\delta D_i]$ will also be a minimal state-space

representation. However Robustness Result I.1 still holds even if $[A_i+\delta A_i, B_i+\delta B_i, C_i+\delta C_i, D_i+\delta D_i]$ is not minimal.

Case II: The transfer functions G_1 and G_2 are specified by coprime matrix fractions representations

Let G_1, G_2 be specified by their coprime factorizations:

$$G_1 = N_{1r}D_{1r}^{-1} \qquad G_2 = D_{2\ell}^{-1}N_{2\ell}$$

where $N_{1r}, D_{1r}, D_{2\ell}, N_{2\ell} \in \mathbb{R}[s]^{n\times n}$ and the pairs (N_{1r},D_{1r}), $(N_{2\ell},D_{2\ell})$ are right-coprime and left-coprime respectively [7,8,9]. Let the allowed parameter perturbations be perturbations in the coefficients of each scalar polynomial entry in the four polynomial matrices N_{1r}, D_{1r}, $D_{2\ell}$, $N_{2\ell}$, without increasing the degree of any scalar polynomial.[††] We know that [10,11] the feedback system S: $(u_1,u_2)\longmapsto(e_1,e_2)$ is exp. stable if and only if its characteristic polynomial

$$\Delta := \det[D_{2\ell}D_{1r} + N_{2\ell}N_{1r}] \tag{9}$$

has all its zeros in $\mathring{\mathbb{C}}_-$. We can also write

$$\Delta = \det D_{2\ell}\cdot\det D_{1r}\cdot\det[I + D_{2\ell}^{-1}N_{2\ell}N_{1r}D_{1r}^{-1}]$$

$$= \det D_{2\ell}\cdot\det D_{1r}\cdot\det[I + G_2G_1] \tag{10}$$

By assumption (2), G_1, G_2 are proper and, for large s, $\det[I + G_2G_1](s) = \det[I + G_2(\infty)G_1(\infty)] + O(1/s)$. Hence degree of Δ = degree of $\det D_{2\ell}$ + degree of $\det D_{1r}$. In view of (10) it follows from the lemma:

Robustness Result II.1.

If the given feedback system S is exp. stable at the nominal data point, then for any sufficiently small allowed perturbation $(\delta N_{1r},\delta D_{1r},\delta D_{2\ell},\delta N_{2\ell})$ which does not increase the degree of $\det D_{2\ell}$ and $\det D_{1r}$, the resulting perturbed system is still exp. stable. □

Remark II.1: Note that for sufficiently small allowed perturbations, $(N_{1r}+\delta N_{1r}, D_{1r}+\delta D_{1r})$ will be right-coprime and $(N_{2\ell}+\delta N_{2\ell}, D_{2\ell}+\delta D_{2\ell})$ will be left-coprime. However Robustness Result II.1 still holds even if they are not coprime.

Let r_i (resp. c_i) be the highest power of s in the ith row (resp. column) of an nxn polynomial matrix D(s). Then degree of det D $\leq \min\{\sum_{i=1}^{n} r_i, \sum_{i=1}^{n} c_i\}$. If the equality holds with $\sum_{i=1}^{n} r_i$, then D is said to be row-proper and if the equality holds with $\sum_{i=1}^{n} c_i$, then D is said to be column-proper [7,8]. Thus if D is either row-proper, or column-proper, then it is easy to see that the degree of det D will never be increased by any allowed perturbation δD. It is well-known that [7,8] for any nxn polynomial matrix D(s) with $\det D(s) \not\equiv 0$, there exist unimodular matrices $U_r(s)$, $U_c(s)$ such that $D(s)\,U_c(s)$ is column-proper and $U_r(s)\,D(s)$ is row-proper. Therefore given a rational matrix, there exist a left-coprime factorization (N_ℓ,D_ℓ) such that D_ℓ is row-proper and a right-coprime factorization (N_r,D_r) such that D_r is column-proper. Summarizing the facts above, we have

Robustness Result II.2.

Suppose that $D_{2\ell}$ is row-proper and D_{1r} is column-proper; if the given feedback system S is exp. stable at the nominal data point, then for any sufficiently small allowed perturbation $(\delta N_{1r},\delta D_{1r},\delta D_{2\ell},\delta N_{2\ell})$ the resulting perturbed system is still exp. stable. □

Remark II.2: The requirement that $D_{2\ell}$ and D_{1r} be resp. row-proper and column-proper is very important, for otherwise the degree of the characteristic polynomial may increase as a result of an arbitrarily small allowed perturbation. This is shown in the following example.

Example: Let $D_{2\ell} = N_{2\ell} = N_{1r} = I$ and

$$D_{1r} = \begin{bmatrix} s+2 & s+1 \\ s+1 & s+3 \end{bmatrix}$$

Hence $\Delta(s) = \det[D_{2\ell}D_{1r}+N_{2\ell}N_{1r}](s) = 5s+11$. Now D_{1r} is not column-proper: consider a small perturbation say, α in the coefficient of s in one of the diagonal elements of D_{1r}. For example, when the (1,1) element becomes $(1+\alpha)s + 2$ we obtain

$$\Delta(s) + \delta\Delta(s) = \alpha s^2 + (5+4\alpha)s + 11$$

which has a zero with positive real part whenever $\alpha < 0$. □

Case III: The transfer functions G_1 and G_2 are specified by the $2n^2$ scalar rational functions which are the elements of G_1 and G_2.

Let the allowed parameter perturbations be the perturbations in the coefficients of the numerator- and denominator-polynomials of each scalar rational function entry in the matrices G_1, G_2 subject to the condition that they

[††] Viewing the matrix entries as polynomials, note that the zero polynomial has degree $-\infty$ and a nonzero constant polynomial has degree 0. Thus the allowed perturbations will not cause a zero in any of the four polynomial matrices to become nonzero.

do not increase the degree of any scalar polynomial.

III.1: Single-input single-output subsystems

Consider the case where $G_i = n_i/d_i$, $i = 1,2$, where n_i, d_i are coprime scalar polynomials, i.e., they are coprime elements of $\mathbb{R}[s]$. Clearly the characteristic polynomial of the feedback system S is $(d_1d_2+n_1n_2)$. Since d_1, d_2 are scalar polynomials, they are both row-proper and column-proper. Thus by Robustness Result II.2, if the given feedback system is exp. stable at the nominal data point, for any sufficiently small allowed perturbation (for Case III), the resulting perturbed system is still exp. stable.

III.2: Multi-input multi-output subsystems

Recall the notations introduced in Case I and

$$\det(sI-A) = \det(sI-A_1)\cdot\det(sI-A_2)\cdot\det(I+G_2G_1)(s) \tag{8}$$

Let

$$p_{o+} := \text{number of } \mathbb{C}_+\text{-zeros of } \det(sI-A_1)\cdot\det(sI-A_2), \text{ counting multiplicities.} \tag{11}$$

Then recall the Graphical Stability Conditions: [12,13] the feedback system S: $(u_1,u_2)\mapsto(e_1,e_2)$ is exp. stable if and only if the Nyquist diagram of $s\mapsto\det[I + G_2(s)G_1(s)]$ — for the contour C which is duly indented to the left at all $j\omega$-axis poles of $\det[I + G_2(s)G_1(s)]$ — does not go through the origin and does encircle the origin p_{o+} times in the counterclockwise sense.

Now suppose that for the nominal parameter values in G_1 and G_2 the Nyquist diagram satisfies the stability conditions above. Consider the effect on the Nyquist diagram of small allowed parameter perturbations in G_1 and G_2. Now, (a) for each $s \in \mathbb{C}$, except at poles, $\det[I + G_2(s)G_1(s)]$ is a continuous function of all the numerator- and denominator-coefficients of G_1 and G_2; (b) the Nyquist diagram of the contour C is a compact curve in $\mathbb{C}$, therefore for any sufficiently small allowed parameter perturbation, the Nyquist diagram will still avoid the origin and encircle it p_{o+} times; (c) for sufficiently small allowed perturbations and for a fixed contour C (indented to the left), any $j\omega$-axis pole of G_1 and/or G_2 that is perturbed will still remain in C_i, the open set in $\mathbb{C}$ enclosed by the indented contour C.

Recall that the order of zero, λ, of $\det(sI-A_i)$ is equal to the McMillan degree of G_i at its pole λ — which we denote by $\Delta(G_i;\lambda)$ [14,15]. Thus

$$p_{o+} = \sum_{i=1,2}\sum_{\lambda}\Delta(G_i;\lambda)$$

where the sums are taken over all the $\mathbb{C}_+$-poles λ of G_1 and G_2, or equivalently, over all the poles λ in C_i of G_1 and G_2. With this in mind we see that some sufficiently small allowed perturbation may change the required number of encirclements in only one way: namely to have

$$p_{o+} \neq \sum_{i=1,2}\sum_{\lambda}\Delta(G_i+\delta G_i;\lambda)$$

where the sums are taken over all the poles λ in C_i of perturbed transfer functions $G_1 + \delta G_1$ and $G_2 + \delta G_2$.

For example, consider

$$G_1(s) = \frac{1}{s-1}\begin{bmatrix}2 & 1\\ 3 & 1.5\end{bmatrix},\quad G_1(s) + \delta G_1(s) = \frac{1}{s-1}\begin{bmatrix}2+\alpha & 1\\ 3 & 1.5\end{bmatrix}$$

then $\Delta(G_1;1) = \operatorname{rank}\begin{bmatrix}2 & 1\\ 3 & 1.5\end{bmatrix} = 1$, [15,p.115], but, for any $\alpha \neq 0$, $\Delta(G_1+\delta G_1;1) = 2$.

Therefore we must formulate our result as follows:

Robustness Result III.1.

If the given feedback system S is exp. stable at the nominal data point, then for any sufficiently small allowed perturbations δG_1, δG_2 which satisfy

$$\sum_{i=1,2}\sum_{\lambda}\Delta(G_i;\lambda) = \sum_{i=1,2}\sum_{\lambda}\Delta(G_i+\delta G_i;\lambda)$$

where the sums are taken over all the poles λ in C_i of the corresponding transfer functions, the resulting perturbed system is still exp. stable. □

In particular, robustness of stability conditions for the feedback system S follows if G_1 and G_2 are both exp. stable.

In the following discussion, we restrict ourselves to simple $\mathbb{C}_+$-poles of G_1 and G_2 because there always exists some allowed perturbation which splits a multiple pole of G_i (i=1 and/or 2) in many ways. To see this suppose that the (1,1) element of G_1 has a third-order pole at s= -1, thus its denominator, d_{11}, has the form

$$d_{11}(s) = (s+1)^3\, p_{11}(s) \quad \text{with } p_{11}(1) \neq 0$$

This triple pole can be split into three simple poles, for example

$$d_{11}(s) + \delta d_{11}(s) = [(s+1)^3 + \varepsilon^3]p_{11}(s)$$

or into a double pole and a simple pole

$$d_{11}(s) + \delta d_{11}(s) = (s+1)^2[(s+1) + \varepsilon]p_{11}(s),$$

etc. More generally, d_{11} has multiple zero(s) if and only if its discriminant $\Delta_{11} = 0$ [16]. Since Δ_{11} is a polynominal in the coefficients of d_{11}, it is clear that multiple zeros are not generic [2].

Consider now the effect of an arbitrary small allowed perturbation on the McMillan degree of G_1 at λ_k, where λ_k is a simple pole with $\operatorname{Re} \lambda_k \geq 0$. Usually only some of the n^2 elements of G_1 have a pole at λ_k and for any sufficiently small allowed perturbation, if the (i,k) element of G_1 has no pole at λ_k, then the (i,k) element of $G_1+\delta G_1$ will still have no pole in a small neighborhood of λ_k. Consequently, only the nonzero elements in the residue matrix R_{ko} of G_1 at λ_k are affected by the allowed perturbations. Thus R_{ko} is a structured matrix in the sense of Shields and Pearson [17,18] i.e., it has a fixed pattern of zero elements. Let ν_1 be the number of coefficients which specify G_1. The generic rank [17,18] of the structured matrix R_{ko} is defined to be the maximal rank that R_{ko} achieves as a function of these ν_1 parameters. R_{ko} does not achieve its generic rank only for parameter values in some proper, closed, nowhere-dense variety $V \subset \mathbb{R}^{\nu_1}$. Consequently, if the rank of the nominal R_{ko} is less than its generic rank, then, for some arbitrarily small allowed perturbation, the rank of R_{ko} will jump to its generic rank. Thus we have the

Robustness Result III.2.

Suppose that all $\mathbb{C}_+$-poles of G_1 and of G_2 are simple and that for each of these poles the nominal residue matrix has a rank equal to its generic rank; if the given feedback system is exp. stable at the nominal data point, then for any sufficiently small allowed perturbation, the resulting perturbed system is still exp. stable.

Furthermore, if the nominal residue matrix of some $\mathbb{C}_+$-pole of either G_1 or G_2, has a rank less than its generic rank, and if the nominal system S is exp. stable, then for some arbitrarily small allowed perturbation, the resulting perturbed system is unstable. □

Remark III.1: The allowed perturbations considered in Case III appear quite reasonable. However, we should be on guard that they might include perturbations that have no physical meaning for the case at hand. For example, this could occur if G_1, instead of being specified by a collection of *n^2 rational functions — the entries of the* matrix G_1 — were specified by a block diagram delineating the interconnections between the subsystems constituting G_1. In that case the appropriate perturbations to consider are not arbitrary perturbations in all the coefficients in the n^2 rational function specifying G_1 but rather perturbations in the parameters specifying the subsystems constituting G_1. This is illustrated by the following example.

Example. Suppose that G_1 consists of a collection of subsystems in series and in parallel (no feedback!) and that only one subsystem has an unstable pole, say at p, with $\operatorname{Re} p \geq 0$. The contribution of that subsystem to G_1 is exhibited on Fig. 2: the ith scalar input of G_1 can only affect the scalar input v of the unstable subsystem through the gain β_i; similarly the unstable subsystem output z is also scalar and is fanned out to the ith output of G_1 through gain γ_i. Clearly the residue of G_1 at p is the dyad (i.e. a rank-one matrix!) $R = \gamma\beta^T$ where $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_n)^T$ $\beta^T = (\beta_1,\beta_2,\ldots,\beta_n)$. Now any small perturbation in the physical parameters (γ_i,β_i), $i = 1,2,\ldots,n$ and p will not change the rank of R because the dyadic structure of R is dictated by the nature of the interconnection of the subsystems constituting G_1, and not by the numerical value of the parameters. Another way of viewing this fact is to say: the only meaningful physical parameters which determine R are the 2n scalars $\gamma_1, \gamma_2,\ldots,\gamma_n, \beta_1,\beta_2,\ldots,\beta_n$. To take the abstract mathematical point of view that R is an array of n^2 real numbers, and therefore n^2 independent perturbations of its parameters are appropriate is mathematical fiction, and not an analysis of the physical system under consideration. □

The lesson of this remark is that before considering robustness, *one should go back to the physical model of the system under consideration and trace out the effect of perturbations of the* physical parameters of the model *on the* coefficients of the mathematical representation. *Only in this way,* the engineer will assure himself that the allowed perturbations he worries about pertain to physical reality and not to mathematical fiction.

Remark III.2: There is another case where perturbations are more restricted than those considered in Case III: in many models Newton's *law dictates a second order pole at the origin —* say in the transfer function from the external forces and the center of mass — . Clearly this second order pole is not subject to perturbations due to measurement or manufacturing errors!

Example: Consider the planar mechanical system described by

$$m\ddot{y}_1 - ky_2 = u_1$$
$$m\ddot{y}_2 = u_2.$$

where (u_1,u_2) is the applied force, (y_1,y_2) is the position of the particle of mass m and the physical parameters are $m > 0$ and $k > 0$. Here

$$G(s) = \begin{bmatrix} m^{-1}s^{-2} & m^{-2}ks^{-4} \\ 0 & m^{-1}s^{-2} \end{bmatrix} = \begin{bmatrix} \alpha s^{-2} & \beta s^{-4} \\ 0 & \alpha s^{-2} \end{bmatrix}$$

where we put $\alpha = m^{-1}$, $\beta = m^{-2}k$. Clearly, in this example the second order pole of G and the fourth order pole of G are not subject to perturbations when the physical parameters α, β are perturbed. In the present case, the McMillan degree at the unstable pole (s=0) is insensitive to small perturbations in the physical parameters. To check this write $G(s) = \sum_{k=0}^{3} R_k s^{k-4}$; then $\Delta(G;0)$ is given by the rank of the Hankel matrix [15]

$$H_o = \begin{bmatrix} R_3 & R_2 & R_1 & R_o \\ R_2 & R_1 & R_o & 0 \\ R_1 & R_o & 0 & 0 \\ R_o & 0 & 0 & 0 \end{bmatrix}$$

where $R_o = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$, $R_2 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$ and the others are zero. It is easily checked that for all allowed physical values (namely, $\alpha > 0$, $\beta > 0$) $\Delta(G;0) = 4$ and also that the generic rank of H_o is 4, [17, p. 211]. Thus if this transfer function G were the forward gain of a closed-loop system with an exponentially stable feedback gain $K(s) = \text{diag}(k_1(s), k_2(s))$ then p_{o+}, the number of required counterclockwise encirclements, would be 4 and it would be robust under any perturbation of the physical parameters α and β. □

Relation between the effect of parameter perturbations in the three cases above.

Consider a given transfer function G(s), one of its minimal state-space realization [A,B,C,D] and one of its right-coprime factorizations (N_r, D_r). Offhand we might think that if some allowed parameter perturbation in Case I leads to the transfer function $G+\delta G$, then there exists some allowed parameter perturbation in Case II which leads to the same transfer function $G+\delta G$, and so on. In other words, we might think that the parameter perturbations in the three cases are, in some sense, equivalent. This is not the case: indeed Remark II.1 shows that for some coprime factorization (N_r, D_r) there is some aribtrarily samll allowed parameter perturbation (of Case *II*) which will increase the degree of the characteristic polynomial, this, however, is impossible under any allowed parameter perturbation in Case I.

Similarly the example below shows that for some G(s) there is some arbitrarily small allowed parameter perturbation of Case I which leads to some transfer function $G+\delta G$ unattainable by any allowed parameter perturbation of Case III.

Example: Consider a transfer function G(s) with the minimal state-space representation [A,B,C,D] where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = C = I_{5x5}, \quad D = 0_{5x5}.$$

Thus, $G(s) = (sI-A)^{-1}$

The (1,2) position minor of $(sI-A)$ is given by

$$\det\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 1 & 0 & 0 & s \end{bmatrix} = 0, \text{ for all } s.$$

Therefore the numerator polynomial of the (2,1) element of G(s) has a degree of $-\infty$. Let δA be a 5x5 matrix which has only one nonzero element α at (3,1) position. Clearly $[\delta A,0,0,0]$ is an allowed parameter perturbation in Case I. The (1,2) position minor of $(sI-A-\delta A)$ is given by

$$\det\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1-\alpha & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 1 & 0 & 0 & s \end{bmatrix} = -\alpha s^2. \text{ Also}$$

$$\det[sI-(A+\delta A)] = s^3(s-1)^2.$$

Therefore the numerator polynomial of (2,1) element of $G+\delta G$ has degree zero. (Note the cancellation!) Due to the increase in degree of the numerator *polynomial of* (2,1) *element*, $G+\delta G$ *is unattainable* from G by any allowed parameter perturbation of Case III.

VI. Conclusion

In this paper we studied the robustness of the exponential stability of continuous-time, linear, time-invariant, lumped, multi-input multi-output feedback systems. We presented robustness results for three types of system representations with corresponding sets of allowed parameter perturbations.

The robustness results above were obtained assuming that the two subsystems G_1 and G_2 have the same number of inputs and outputs, all the results can be extended, after simple modifications, to the case where the number of inputs and outputs of each subsystems are different.

Also since all the arguments used are purely algebraic and are based on simple properties of rational functions and polynomials, all the results above apply equally *well to* the discrete-time case except that in Robustness Result III.1, C should be interpreted as the unit circle with inward indentation to avoid the poles of G_1 and G_2, lying on the unit circle, and C_1 should be interpreted as the "outside" of C (more precisely the unbounded connected component of $\mathbb{C}$-C).

VII. Acknowledgement

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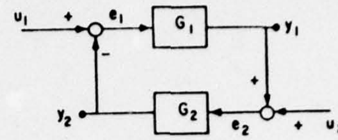

Fig. 1. Multi-input multi-output feedback system under consideration.

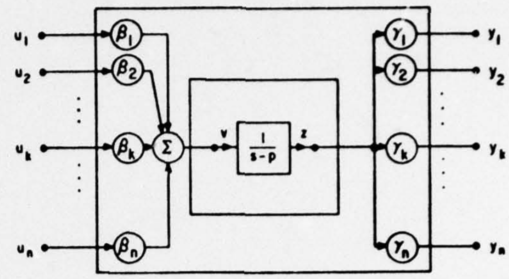

Fig. 2. Example of a system whose transfer function residue matrix has rank one for all physical values of the parameters β_i and γ_i.

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The robustness of stability conditions for linear time-invariant feedback syst is examined assuming three different types of representations: state-space rep sentation, coprime matrix fraction representation, and transfer function repre sentation. We stress the importance of certain details of the representation used and, even more, the importance of making sure that the allowed perturbati be relevant to the physical situation under study.

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